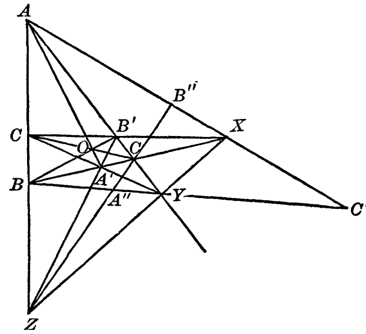
I. Solution by William Hoover, Columbus, Ohio.

Take ABC as the triangle of reference, and the trilinear coördinates of the vertices A, B, C to be $(\alpha_2, 0, 0)$; $(0, \beta_2, 0)$; $(0, 0, \gamma_2)$, and those of O, $(\alpha_1, \beta_1, \gamma_1)$; then the equation of AO is $\gamma_1\beta - \beta_1\gamma = 0$.

The coördinates of A' will be proportional to $(0, \beta_1, \gamma_1)$, and, by symmetry, those of B' to $(\alpha_1, 0, \gamma_1)$, and of C' to $(\alpha_1, \beta_1, 0)$.

The equation of $A'B'$ is $\beta_1\gamma_1\alpha + \alpha_1\gamma_1\beta - \alpha_1\beta_1\gamma = 0$.



The point of intersection of AB and $A'B'$, or Z, has coördinates proportional to $(-\alpha_1, \beta_1, 0)$; those of the intersection BC and $B'C'$, or X, to $(0, -\beta_1, \gamma_1)$; and of CA and $C'A'$, or Y, to $(\alpha_1, 0, -\gamma_1)$; and X, Y, Z are collinear, since

$$\begin{vmatrix} 0, & -\beta_1, & \gamma_1 \\ \alpha_1, & 0, & -\gamma_1 \\ -\alpha_1, & \beta_1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & -1, & 1 \\ 1, & 0, & -1 \\ -1, & 1, & 0 \end{vmatrix}$$

$$= \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & 0, & 1 \\ 1, & -1, & -1 \\ -1, & 1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 1, & -1 \\ -1, & 1 \end{vmatrix} = 0.$$

The equation of AX is $\gamma_1\beta + \beta_1\gamma = 0$, and those of BY and CZ respectively $\gamma_1\alpha + \alpha_1\gamma = 0$, and $\beta_1\alpha + \alpha_1\beta = 0$.

The coördinates of C'', the point of intersection of AX and BY, are proportional to $(\alpha_1, \beta_1, \gamma_1)$; of BY and CZ, to $(\alpha_1, \beta_1, \gamma_1)$; and of CZ and AX to $(\alpha_1, \beta_1, \gamma_1)$, the latter two points being A'', B'' respectively.

It is evident now that the equations to AA'', BB'', CC'' are the same as those of AO, BO, CO, in order, the first three lines then passing through O.

II. Solution by H. L. Olson, Chicago, Illinois.

I shall amplify this theorem by proving that the lines AA'', BB'', CC'' are identical with the lines AA', BB', CC' respectively, and hence intersect in the point O. In the triangles $BC'Y$ and $CB'Z$, the lines BC, $C'B'$, and YZ, joining corresponding vertices, meet at X, and hence the points A, A', A'', in which corresponding sides meet, are collinear. Similarly B, B', B'' are collinear; also C, C', C''. Hence the lines AA'', BB'', CC'' meet at O.

Also solved by A. Pelletier and the Proposer.

2716 [June, 1918]. Proposed by Clifford N. Mills, Brookings, South Dakota.

To a passenger in a train moving at the rate of 40 miles per hour, the rain appears to be rushing downward and towards him at an angle of 20 degrees with the horizontal. If the rain is actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

Solution by E. H. Worthington, Elkins Park, Pa.

A velocity of 40 miles per hour is the same as $58\frac{2}{3}$ feet per second. If v is the velocity of the raindrop, we have $v = 58\frac{2}{3}\tan 20° = 58\frac{2}{3} \times 0.364$ feet per sec. $= 21.35$ feet per second.

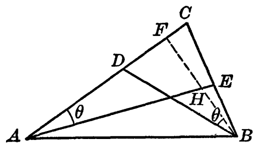
Also solved by H. E. Carleton, A. M. Harding, H. L. Olson, A. Pelletier and J. B. Reynolds.

2735 [December, 1918]. Proposed by H. B. Phillips, Massachusetts Institute of Technology.

If two lines AE and BD, drawn from the vertices A and B of a triangle to the opposite sides, divide the angles A and B so that the parts of A are respectively less than the corresponding parts of B, then AE is greater than BD.

SOLUTION BY THE PROPOSER.

Let C be the third vertex. By hypothesis $CAE < CBD$ and $BAE < ABD$. A point F can then be found on CD such that $DBF = CAE$. Let BF cut AE in H. AHF and BDF are similar, having equal angles at A and B and the same angle at F. Therefore,



$$AH : BD = AF : BF. \tag{1}$$

Also, since $ABD > BAE$, $ABF > BAF$. Consequently,

$$AF > BF. \tag{2}$$

From (1) and (2), it follows that $AH > BD$. Therefore, $AE > AH > BD$, which was to be proved.

COROLLARY: *If AE and BD are equal and divide their angles in the same ratio, the triangle is isosceles.*

For, if the angles A and B were not equal the parts of one would be respectively less than the corresponding parts of the other and AE and BD would be unequal, which is contrary to hypothesis.

In particular, if the bisectors of two angles of a triangle are equal, the triangle is isosceles.

2736 [December, 1918]. Proposed by M. COHEN, Freshman, Johns Hopkins University.

Prove by elementary geometry that the orthocenter, the centroid, and the circumcenter of a triangle lie on a line (the Euler line), and that the centroid lies between the other two and is twice as far from the orthocenter as from the circumcenter.

SOLUTION BY J. L. RILEY, Stephenville, Texas.

Let ABC be the triangle under consideration; O and G the circumcenter and centroid, BE and CF perpendicular, respectively, to AC and AB. Let mid-point of AC be B'.

Produce OG to meet the altitude BE at K. The triangles OGB' and KGB are similar, for OB' is parallel to BK, since each is perpendicular to AC. Then $OG : GK = B'G : GB = 1 : 2$ and hence, $GK = 2\,OG$.

If OG is produced to meet the altitude CF at K', it follows in the same way that $GK' = 2\,OG$. Therefore, $GK' = GK$ and K' coincides with K. Hence BE and CF meet at K and K is the orthocenter. Hence, circumcenter, centroid, and orthocenter lie on the same line.

Also solved by H. L. OLSON, C. P. SOUSLEY, and the Proposer.

2737 [January, 1919]. Proposed by C. N. SCHMALL, New York City.

Employing Maclaurin's theorem, or otherwise, expand the following three functions (1) $e^{\tan^{-1}x}$ as far as x^6; (2) $e^{\sin x}$ as far as x^8; and (3) $\tan x$ as far as x^9.

SOLUTION BY ELMER LATSHAW, West Philadelphia, Pennsylvania.

The successive differentiation required by Maclaurin's theorem in the development of the given functions is long and laborious, but the required developments may be obtained by comparing the derivative of the function with the function itself.

Assume

$$e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_6x^6 + \cdots. \tag{1}$$

Differentiating both sides of (1),

$$\begin{aligned} e^{\tan^{-1}x}\frac{1}{1+x^2} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 6a_6x^5 + \cdots \\ &= (a_0 + a_1x + a_2x^2 + \cdots a_6x^6 + \cdots)(1 - x^2 + x^4 - x^6 + \cdots). \end{aligned}$$

Equating coefficients of like powers of x,

$$a_1 = a_0,\quad 2a_2 = a_1,\quad 3a_3 = a_2 - a_0,\quad 4a_4 = a_3 - a_1,\quad 5a_5 = a_4 - a_2 + a_0,\quad 6a_6 = a_5 - a_3 + a_1.$$

Equation (1) by making $x = 0$ gives $a_0 = 1$ and the preceding equations give

$$a_1 = 1,\qquad a_2 = \tfrac{1}{2},\qquad a_3 = -\tfrac{1}{6},\qquad a_4 = -\tfrac{7}{24},\qquad a_5 = \tfrac{1}{24},\qquad a_6 = \tfrac{29}{144}.$$

Hence,

$$e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + \frac{x^5}{24} + \frac{29x^6}{144} - \cdots.$$